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**Matrix-based approach to electrodynamics in media**

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The Riemann – Silberstein – Majorana – Oppenheimer approach to the Maxwell electrodynamics in presence of electrical sources and arbitrary media is investigated within the matrix formalism. The symmetry of the matrix Maxwell equation under transformations of the complex rotation group  $SO(3.C)$  is demonstrated explicitly. In vacuum case, the matrix form includes four real  $4 \times 4$  matrices  $\alpha^b$ . In presence of media matrix form requires two sets of  $4 \times 4$  matrices,  $\alpha^b$  and  $\beta^b$  – simple and symmetrical realization of which is given. Relation of  $\alpha^b$  and  $\beta^b$  to the Dirac matrices in spinor basis is found. Minkowski constitutive relations in case of any linear media are given in a short algebraic form based on the use of complex 3-vector fields and complex orthogonal rotations from  $SO(3.C)$  group. The matrix complex formulation in the Esposito's form, based on the use of two electromagnetic 4-vectors,  $e^\alpha(x) = u_\beta F^{\alpha\beta}(x)$ ,  $b^\alpha(x) = u_\beta \tilde{F}^{\alpha\beta}(x)$  is studied and discussed. It is argued that Esposito form is achieved through the use of a trivial identity  $I = U^{-1}(u)U(u)$  in the Maxwell equation.

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## 1 Introduction

Special relativity arose from study of the symmetry properties of the Maxwell equations with respect to motion of reference frames: Lorentz [2], Poincaré [3], Einstein [4]. Naturally, an analysis of the Maxwell equations with respect to Lorentz transformations was the first objects of relativity theory: Minkowski [5], Silberstein [6]-[7], Marcolongo [8], Bateman [9], and Lanczos [10], Gordon [11], Mandel'stam – Tamm [12]-[13]-[14].

After discovering the relativistic equation for a particle with spin 1/2 – Dirac [15] – much work was done to study spinor and vectors within the Lorentz group theory: Möglich [16], Ivanenko – Landau [17], Neumann [18], van der Waerden [19], Juvet [20]. As was shown, any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. A spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [21], also see Rumer [28]. In 1931, Majorana [23] and Oppenheimer [22] proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a complex 3-vector wave function satisfying the massless Dirac-like equations. Before Majorana and Oppenheimer, the most crucial steps were made by Silberstein [6], he showed the possibility to have formulated Maxwell equation in terms of complex 3-vector entities. Silberstein in his second paper [7] writes that the complex form of Maxwell equations has been known before; he refers there to the second volume of the lecture notes on the differential equations of mathematical

physics by B. Riemann that were edited and published by H. Weber in 1901 [1]. This not widely used fact is noted by Bialynicki-Birula [103]).

Maxwell equations in the matrix Dirac-like form considered during long time by many authors, the interest to the Majorana-Oppenheimer formulation of electrodynamics has grown in recent years:

Luis de Broglie [24]-[25]-[30]-[36], Petiau [26], Proca [27]- [44], Duffin [29], Kemmer [31]-[41]-[61], Bhabha [32], Belinfante [33]-[34], Taub [35], Sakata – Taketani [37], Schrödinger [39]-[40], Heitler [42], [45]-[46], Mercier [47], Imaeda [48], Fujiwara [50], Ohmura [51], Borgardt [52]-[59], Fedorov [53], Kuohsien [54], Bludman [55], Good [56], Moses [57]-[60]-[76], Lomont [58], Bogush – Fedorov [64], Sachs – Schwebel [66], Ellis [68], Oliver [70], Beckers – Piroette [71], Casanova [72], Carmeli [73], Bogush [74], Lord [75], Weingarten [77], Mignani – Recami – Baldo [78], Newman [79], [80], [82], Edmonds [83], Silveira [86], Jena – Naik – Pradhan [87], Venuri [88], Chow [89], Fushchich – Nikitin [90], Cook [92]-[93], Giannetto [96], – Yépez, Brito – Vargas [97], Kidd – Ardini – Anton [98], Recami [99], Krivsky – Simulik [101], Inagaki [102], Bialynicki-Birula [103]-[104]-[128], Sipe [105], [106], Esposito [108], Dvoeglazov [109] (see a big list of relevant references therein)-[111], Gersten [107], Kanatchikov [110], Gsponer [112], [113]-[118]-[114]-[115]-[119]- [124]-[125]-[126]-[127], Donev – Tashkova. [121]-[122]-[123].

Our treatment will be with a quite definite accent: the main attention is given to technical aspect of classical electrodynamics based on the theory of rotation complex group  $SO(3,C)$  (isomorphic to the Lorentz group – see Kurşunoğlu [62], Macfarlane cite[65]-[69], Fedorov [84]).

## 2 Complex matrix form of Maxwell theory in vacuum

Let us start with Maxwell equations in a uniform  $(\epsilon, \mu)$ -media in presence of external sources [38]-[63]-[82]:

$$\begin{aligned} (F^{ab}) \quad \text{div } c\mathbf{B} &= 0, \quad \text{rot } \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct}, \\ (H^{ab}) \quad \text{div } \mathbf{E} &= \frac{\rho}{\epsilon\epsilon_0}, \quad \text{rot } c\mathbf{B} = \mu\mu_0 c\mathbf{J} + \epsilon\mu \frac{\partial \mathbf{E}}{\partial ct}. \end{aligned} \quad (1)$$

With the use of usual notation for current 4-vector  $j^a = (\rho, \mathbf{J}/c)$ ,  $c^2 = 1/\epsilon_0\mu_0$ , eqs. (1) read (first, consider the vacuum case):

$$\begin{aligned} \text{div } c\mathbf{B} &= 0, \quad \text{rot } \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct}, \\ \text{div } \mathbf{E} &= \frac{\rho}{\epsilon_0}, \quad \text{rot } c\mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial ct}. \end{aligned} \quad (2)$$

Let us introduce 3-dimensional complex vector

$$\psi^k = E^k + icB^k, \quad (3)$$

with the help of which the above equations can be combined into (see Silberstein [6]-[7], Bateman [9], Majorana [23], Oppenheimer [22], and many others)

$$\begin{aligned} \partial_1 \Psi^1 + \partial_2 \Psi^0 + \partial_3 \Psi^3 &= j^0/\epsilon_0, \quad -i\partial_0 \psi^1 + (\partial_2 \psi^3 - \partial_3 \psi^2) = i j^1/\epsilon_0, \\ -i\partial_0 \psi^2 + (\partial_3 \psi^1 - \partial_1 \psi^3) &= i j^2/\epsilon_0, \quad -i\partial_0 \psi^3 + (\partial_1 \psi^2 - \partial_2 \psi^1) = i j^3/\epsilon_0, \end{aligned} \quad (4)$$

where  $x_0 = ct$ ,  $\partial_0 = c \partial_t$ . These four relations can be rewritten in a matrix form using a 4-dimensional column  $\psi$  with one additional zero-element (Fuschich – Nikitin [90]):

$$(-i\alpha^0\partial_0 + \alpha^j\partial_j)\Psi = J, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}, \quad \alpha^0 = \begin{vmatrix} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{vmatrix},$$

$$\alpha^1 = \begin{vmatrix} b_0 & 1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & -1 \\ b_3 & 0 & 1 & 0 \end{vmatrix}, \alpha^2 = \begin{vmatrix} c_0 & 0 & 1 & 0 \\ c_1 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 \\ c_3 & -1 & 0 & 0 \end{vmatrix}, \alpha^3 = \begin{vmatrix} d_0 & 0 & 0 & 1 \\ d_1 & 0 & -1 & 0 \\ d_2 & 1 & 0 & 0 \\ d_3 & 0 & 0 & 0 \end{vmatrix}. \quad (5)$$

Here, there arise four matrices, in which numerical parameters  $a_k, b_k, c_k, d_k$  may be arbitrary. Our choice for the matrix form of eight Maxwell equations is the following:

$$(-i\partial_0 + \alpha^j\partial_j)\Psi = J, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}, \quad J = \frac{1}{\epsilon_0} \begin{vmatrix} j^0 \\ i j^1 \\ i j^2 \\ i j^3 \end{vmatrix},$$

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$

$$(\alpha^1)^2 = -I, \quad (\alpha^2)^2 = -I, \quad (\alpha^3)^2 = -I,$$

$$\alpha^1\alpha^2 = -\alpha^2\alpha^1 = \alpha^3, \quad \alpha^2\alpha^3 = -\alpha^3\alpha^2 = \alpha^1, \quad \alpha^3\alpha^1 = -\alpha^1\alpha^3 = \alpha^2. \quad (6)$$

Now let us consider the problem of relativistic invariance of this equation. The lack of manifest invariance of 3-vector complex form of Maxwell theory has been intensively discussed in various aspects: for instance, see Esposito [108], Ivezić [113]-[118]-[114]-[115]-[119]- [124]-[125]-[126]-[127]).

Arbitrary Lorentz transformation over the function  $\Psi$  is given by (take notice that one may introduce four undefined parameters  $s_0, \dots, s_3$ , but we will take  $s_0 = 1, s_j = 0$ )

$$S = \begin{vmatrix} s_0 & 0 & 0 & 0 \\ s_1 & \cdot & \cdot & \cdot \\ s_2 & \cdot & O(k) & \cdot \\ s_3 & \cdot & \cdot & \cdot \end{vmatrix}, \quad \Psi' = S\Psi, \quad \Psi = S^{-1}\Psi', \quad (7)$$

where  $O(k)$  stands for a  $(3 \times 3)$ -rotation complex matrix from  $SO(3, C)$ , isomorphic to the Lorentz group – more detail see in [84] and below in the present text. Equation for a primed function  $\Psi'$  is

$$(-i\partial_0 + S\alpha^j S^{-1}\partial_j) \Psi' = S J.$$

When working with matrices  $\alpha^j$  we will use vectors  $\mathbf{e}_i$  and  $(3 \times 3)$ -matrices  $\tau_i$ , then the structure  $S\alpha^j S^{-1}$  is

$$S\alpha^j S^{-1} = \begin{vmatrix} 0 & \mathbf{e}_j O^{-1}(k) \\ -O(k)\mathbf{e}_j^t & O(k)\tau_j O^{-1}(k) \end{vmatrix} = \alpha^m O_{mj}(k). \quad (8)$$

Therefore, the Maxwell equation gives

$$(-i\partial_0 + \alpha^m \partial'_m)\Psi' = SJ, \quad O_{mj}\partial_j = \partial'_m. \quad (9)$$

Now, one should give special attention to the following: the symmetry properties given by (9) look satisfactory only at real values of parameter  $a$  – in this case it describes symmetry of the Maxwell equations under Euclidean rotations. However, if the values of  $a$  are imaginary the above transformation  $S$  gives a Lorentzian boost; for instance, in the plane 0 – 3 the boost is

$$a = ib, \quad S(a = ib) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & ch\,b & -ish\,b & 0 \\ 0 & ish\,b & ch\,b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (10)$$

and the formulas (8) will take the form

$$\begin{aligned} S\alpha^1 S^{-1} &= ch\,b\,\alpha^1 + ish\,b\,\alpha^2, \\ S\alpha^2 S^{-1} &= -ish\,b\,\alpha^1 + ch\,b\,\alpha^2, \quad S\alpha^3 S^{-1} = \alpha^3. \end{aligned} \quad (11)$$

Correspondingly, the Maxwell matrix equation after transformation (10)-(11) will look asymmetric

$$\begin{aligned} &[(-i\partial_0 + \alpha^3\partial_3) + (ch\,b\,\alpha^1 + ish\,b\,\alpha^2)\partial_2 + \\ &+ (-ish\,b\,\alpha^1 + ch\,b\,\alpha^2)\partial_3] \Psi' = SJ. \end{aligned} \quad (12)$$

However, one can note an identity

$$\begin{aligned} &(ch\,b - ish\,b\,\alpha^3)(-i\partial_0 + \alpha^3\partial_3) = \\ &= -i(ch\,b\,\partial_0 - sh\,b\,\partial_3) + \alpha^3(-sh\,b\,\partial_0 + ch\,b\,\partial_3) = -i\partial'_0 + \alpha^3\partial'_3, \end{aligned} \quad (13)$$

where derivatives are changed in accordance with the Lorentzian boost rule:

$$ch\,b\,\partial_0 - sh\,b\,\partial_3 = \partial'_0, \quad -sh\,b\,\partial_0 + ch\,b\,\partial_3 = \partial'_3.$$

Therefore. it remains to determine the action of the operator (we introduce special notation for it,  $\Delta$ )

$$\Delta = ch\,b - ish\,b\,\alpha^3 \quad (14)$$

on two other terms in eq. (12) – one might expect two relations:

$$\begin{aligned} &(ch\,b - ish\,b\,\alpha^3)(ch\,b\,\alpha^1 + ish\,b\,\alpha^2) = \alpha^2, \\ &(ch\,b - ish\,b\,\alpha^3)(-ish\,b\,\alpha^1 + ch\,b\,\alpha^2) = \alpha^3. \end{aligned} \quad (15)$$

As easily verified they hold indeed. We should calculate the term

$$\Delta S J = \begin{vmatrix} ch\,b\,j^0 + sh\,b\,j^3 \\ i\,j^1 \\ i\,j^2 \\ i(sh\,b\,j^0 + ch\,b\,j^3) \end{vmatrix}; \quad (16)$$

the right-hand side of (16) is what we need.

Thus, the symmetry of the matrix Maxwell equation under the Lorentzian boost in the plane 0 – 3 is described by relations:

$$\Delta(b) (-i\partial_0 + S\alpha^j S^{-1}\partial_j) \Psi' = \Delta S J \equiv J' , \quad (-i\partial'_0 + \alpha^j \partial'_j) \Psi' = J' ,$$

$$S(b) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & ch\,b & -ish\,b & 0 \\ 0 & ish\,b & ch\,b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} , \quad \Delta(b) = ch\,b - ish\,b\,\alpha^3 . \quad (17)$$

For the general case, one can think that for an arbitrary oriented boost the operator  $\Delta$  should be of the form:

$$\Delta = \Delta_\alpha = ch\,b - i\,sh\,b\,n_j\,\alpha^j .$$

To verify this, one should obtain mathematical description of that general boost. We will start with the known parametrization of the real 3-dimension group [84])

$$O(c) = I + 2 [ c_0 \vec{c}^\times + (\vec{c}^\times)^2 ] , \quad (\vec{c}^\times)_{kl} = -\epsilon_{klj} a_j ,$$

$$O(c) = \begin{vmatrix} 1 - 2(c_2^2 + c_3^2) & -2c_0c_3 + 2c_1c_2 & +2c_0c_2 + 2c_1c_3 \\ +2c_0c_3 + 2c_1c_2 & 1 - 2(c_3^2 + c_1^2) & -2c_0c_1 + 2c_2c_3 \\ -2c_0c_2 + 2c_1c_3 & +2c_0c_1 + 2c_2c_3 & 1 - 2(c_1^2 + c_2^2) \end{vmatrix} . \quad (18)$$

Transition to a general boost is achieved by the change

$$c_0 \Rightarrow ch\,\frac{b}{2} , \quad c_j \Rightarrow i\,sh\,\frac{b}{2}\,n_j , \quad n_j n_j = 1 , \quad O(ib, \mathbf{n}) =$$

$$= \begin{vmatrix} 1 + 2sh^2(b/2)(n_2^2 + n_3^2) & -ish\,b\,n_3 - 2sh^2(b/2)n_1n_2 & ish\,b\,n_2 - 2sh^2(b/2)n_1n_3 \\ ish\,b\,n_3 - 2sh^2(b/2)n_1n_2 & 1 + 2sh^2(b/2)(n_3^2 + n_1^2) & -ish\,b\,n_1 - 2sh^2(b/2)n_2n_3 \\ -ish\,b\,n_2 - 2sh^2(b/2)n_1n_3 & ish\,b\,n_1 - 2sh^2(b/2)n_2n_3 & 1 + 2sh^2(b/2)(n_1^2 + n_2^2) \end{vmatrix} \quad (19)$$

from this taking in mind the elementary formula  $1 - ch\,b = -2\,sh^2(b/2)$  , we arrive at

$$O(b, \mathbf{n}) = \begin{vmatrix} 1 - (1 - ch\,b)(n_2^2 + n_3^2) & -ish\,b\,n_3 + (1 - ch\,b)n_1n_2 & ish\,b\,n_2 + (1 - ch\,b)n_1n_3 \\ ish\,b\,n_3 + (1 - ch\,b)n_1n_2 & 1 - (1 - ch\,b)(n_3^2 + n_1^2) & -ish\,b\,n_1 + (1 - ch\,b)n_2n_3 \\ -ish\,b\,n_2 + (1 - ch\,b)n_1n_3 & ish\,b\,n_1 + (1 - ch\,b)n_2n_3 & 1 - (1 - ch\,b)(n_1^2 + n_2^2) \end{vmatrix} . \quad (20)$$

We need to examine relation

$$\Delta(b, \mathbf{n}) (-i\partial_0 + \alpha^i O_{ij}(b, \mathbf{n})\partial_j) \Psi' = \Delta(b, \mathbf{n}) S J ,$$

$$\Delta = ch\,b - ish\,b\,n_1\alpha^1 - ish\,b\,n_2\alpha^2 - ish\,b\,n_3\alpha^3 .$$

After rather long calculation we can indeed prove the general statement: the matrix Maxwell equation is invariant under an arbitrary Lorentzian boost:

$$\begin{aligned}
\Delta(-i \partial_0 + S \alpha^i S^{-1} \partial_i) S \Psi &= \Delta S J \implies (-i \partial'_0 + \alpha^i \partial'_i) \Psi' = J' , \\
S(ib, \mathbf{n}) &= \begin{vmatrix} 1 & 0 \\ 0 & O(ib, \mathbf{n}) \end{vmatrix}, \\
t' &= ch \beta t + sh \beta \mathbf{n} \cdot \mathbf{x}, \quad \mathbf{x}' = +\mathbf{n} sh \beta t + \mathbf{x} + (ch \beta - 1) \mathbf{n} (\mathbf{n} \cdot \mathbf{x}), \\
\partial'_0 &= ch b \partial_0 - sh b (\mathbf{n} \cdot \nabla), \quad \nabla' = -sh b \mathbf{n} \partial_0 + [\nabla + (ch b - 1) \mathbf{n} (\mathbf{n} \cdot \nabla)], \\
j'^0 &= ch b j^0 + sh b (\mathbf{n} \cdot \mathbf{j}), \quad \mathbf{j}' = +sh b \mathbf{n} j^0 + \mathbf{j} + (ch b - 1) \mathbf{n} (\mathbf{n} \cdot \mathbf{j}).
\end{aligned} \tag{21}$$

Invariance of the matrix equation under Euclidean rotations is achieved in a simpler way:

$$\begin{aligned}
(-i \partial_0 + \alpha^i \partial_i) \Psi &= J, \quad S(a, \mathbf{n}) = \begin{vmatrix} 1 & 0 \\ 0 & O(a, \mathbf{n}) \end{vmatrix}, \quad t' = t, \quad \mathbf{x}' = R(a) \mathbf{x}, \\
(-i \partial_0 + S \alpha^i S^{-1} \partial_i) S \Psi &= S J \implies (-i \partial'_0 + \alpha^i \partial'_i) \Psi' = J', \\
\partial'_0 &= \partial_0, \quad \nabla' = R(a, -\mathbf{n}) \nabla, \quad j'^0 = j^0, \quad \mathbf{j}' = R(a, \mathbf{n}) \mathbf{j}.
\end{aligned} \tag{22}$$

### 3 On Maxwell equations in a uniform media, modified Lorentz symmetry

Let us start with Maxwell equations in a uniform media:

$$\begin{aligned}
\operatorname{div} c \mathbf{B} &= 0, \quad \operatorname{rot} \mathbf{E} = -\frac{\partial c \mathbf{B}}{\partial ct}, \\
\operatorname{div} \mathbf{E} &= \frac{\rho}{\epsilon \epsilon_0}, \quad \operatorname{rot} c \mathbf{B} = \mu \mu_0 c \mathbf{J} + \epsilon \mu \frac{\partial \mathbf{E}}{\partial ct}.
\end{aligned} \tag{23}$$

The coefficient  $\epsilon \mu$  can be factorized as follows

$$\epsilon \mu = \sqrt{\epsilon \mu} \sqrt{\epsilon \mu} = \frac{1}{k^2}, \quad c' = \frac{1}{\sqrt{\epsilon_0 \epsilon \mu_0 \mu}} = k c;$$

introducing the variables

$$x^a = (x^0 = kct, \quad x^i), \quad j^a = (j^0 = \rho, \mathbf{j} = \frac{\mathbf{J}}{kc}), \tag{24}$$

eqs. (23) can be rewritten as

$$\begin{aligned}
\operatorname{div} kc \mathbf{B} &= 0, \quad \operatorname{rot} \mathbf{E} = -\frac{\partial kc \mathbf{B}}{\partial x^0}, \\
\operatorname{div} \mathbf{E} &= \frac{1}{\epsilon \epsilon_0} j^0, \quad \operatorname{rot} kc \mathbf{B} = \frac{1}{\epsilon \epsilon_0} \mathbf{j} + \frac{\partial \mathbf{E}}{\partial x^0},
\end{aligned} \tag{25}$$

Equations (25) formally differ from eqs. (2) only in two formal changes:  $c \implies c' = kc$  and  $\epsilon_0 \implies \epsilon_0 \epsilon$ ; therefore, all analysis performed in Section 2 is applicable here:

$$(-i \partial_0 + \alpha^i \partial_i) \Psi = J, \quad \psi^k = E^k + i c' B^k. \tag{26}$$

with the same old matrices involved. The given matrix form of the Maxwell theory in a uniform media proves existence of symmetry of the theory under a modified Lorentz group (Rosen [49]) in which instead of vacuum speed of light we are to use the modified speed:

$$c' = kc, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}, \quad k = \frac{1}{\sqrt{\epsilon \mu}}. \quad (27)$$

## 4 On the matrix form of Maxwell-Minkowski electrodynamics in media

In agreement with Minkowski approach [5], in presence of a uniform media we should introduce two electromagnetic tensors  $F^{ab}$  and  $H^{ab}$  that transform independently under the Lorentz group. At this, the known constitutive (or material) relations change their form in the moving reference frame (for instance, see [91]).

In the rest media, the Maxwell equations are

$$\begin{aligned} \operatorname{div} \mathbf{B} &= 0, & \operatorname{rot} \mathbf{E} &= -\frac{\partial c\mathbf{B}}{\partial ct}, \\ \operatorname{div} \mathbf{D} &= \rho, & \operatorname{rot} \frac{\mathbf{H}}{c} &= \frac{\mathbf{J}}{c} + \frac{\partial \mathbf{D}}{\partial ct}, \end{aligned} \quad (28)$$

with some constitutive relations.

Quantities with simple transformation laws under the Lorentz group are

$$\mathbf{f} = \mathbf{E} + ic\mathbf{B}, \quad \mathbf{h} = \frac{1}{\epsilon_0} (\mathbf{D} + i\mathbf{H}/c), \quad j^a = (j^0 = \rho, \mathbf{j} = \mathbf{J}/c); \quad (29)$$

where  $\mathbf{f}, \mathbf{h}$  are complex 3-vector under complex orthogonal group  $SO(3.C)$ , the latter is isomorphic to the Lorentz group. One can combine eqs. (28) into following ones

$$\begin{aligned} \operatorname{div} \left( \frac{\mathbf{D}}{\epsilon_0} + i c\mathbf{B} \right) &= \frac{1}{\epsilon_0} \rho, \\ -i\partial_0 \left( \frac{\mathbf{D}}{\epsilon_0} + i c\mathbf{B} \right) + \operatorname{rot} \left( \mathbf{E} + i \frac{\mathbf{H}}{\epsilon_0} \right) &= \frac{i}{\epsilon_0} \mathbf{j}. \end{aligned} \quad (30)$$

Eqs. (30) can be rewritten in the form

$$\begin{aligned} \operatorname{div} \left( \frac{\mathbf{h} + \mathbf{h}^*}{2} + \frac{\mathbf{f} - \mathbf{f}^*}{2} \right) &= \frac{1}{\epsilon_0} \rho, \\ -i\partial_0 \left( \frac{\mathbf{h} + \mathbf{h}^*}{2} + \frac{\mathbf{f} - \mathbf{f}^*}{2} \right) + \operatorname{rot} \left( \frac{\mathbf{f} + \mathbf{f}^*}{2} + \frac{\mathbf{h} - \mathbf{h}^*}{2} \right) &= \frac{i}{\epsilon_0} \mathbf{j}. \end{aligned} \quad (31)$$

It has a sense to define two quantities:

$$\mathbf{M} = \frac{\mathbf{h} + \mathbf{f}}{2}, \quad \mathbf{N} = \frac{\mathbf{h}^* - \mathbf{f}^*}{2},$$

which are different 3-vectors under the group  $SO(3.C)$ :

$$\mathbf{M}' = O \mathbf{M}, \quad \mathbf{N}' = O^* \mathbf{N}.$$

With respect to Euclidean rotations, the identity  $O^* = O$  holds; whereas for Lorentzian boosts we have quite other identity  $O^* = O^{-1}$ . In terms of  $\mathbf{M}, \mathbf{N}$ , eqs. (31) look

$$\begin{aligned} \operatorname{div} \mathbf{M} + \operatorname{div} \mathbf{N} &= \frac{1}{\epsilon_0} \rho , \\ -i\partial_0 \mathbf{M} + \operatorname{rot} \mathbf{M} - i\partial_0 \mathbf{N} - \operatorname{rot} \mathbf{N} &= \frac{i}{\epsilon_0} \mathbf{j} , \end{aligned} \quad (32)$$

or in a matrix form

$$\begin{aligned} &(-i\partial_0 + \alpha^i \partial_i) M + (-i\partial_0 + \beta^i \partial_i) N = J , \\ M &= \begin{vmatrix} 0 \\ \mathbf{M} \end{vmatrix} , \quad N = \begin{vmatrix} 0 \\ \mathbf{N} \end{vmatrix} , \quad J = \frac{1}{\epsilon_0} \begin{vmatrix} \rho \\ i \mathbf{j} \end{vmatrix} . \end{aligned} \quad (33)$$

The matrices  $\alpha^i$  and  $\beta^i$  are taken in the form

$$\begin{aligned} \alpha^1 &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix} , & \alpha^2 &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} , & \alpha^3 &= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} , \\ \beta^1 &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} , & \beta^2 &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} , & \beta^3 &= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} . \end{aligned}$$

All of them after squaring give  $-I$ , and  $\alpha_i$  commute with  $\beta_j$ .

## 5 Minkowski constitutive relations in a complex 3-vector form

Let us examine how the constitutive relations for an uniform media behave under the Lorentz transformations. One should start with these relation in the rest media

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E} , \quad \frac{\mathbf{H}}{c} = \frac{1}{\mu_0 \mu} \frac{1}{c^2} c \mathbf{B} = \frac{\epsilon_0}{\mu} c \mathbf{B} . \quad (34)$$

Eqs. (34) can be rewritten as

$$\frac{\mathbf{h} + \mathbf{h}^*}{2} = \epsilon \frac{\mathbf{f} + \mathbf{f}^*}{2} , \quad \frac{\mathbf{h} - \mathbf{h}^*}{2} = \frac{1}{\mu} \frac{\mathbf{f} - \mathbf{f}^*}{2} ; \quad (35)$$

from whence it follows

$$2\mathbf{h} = \left(\epsilon + \frac{1}{\mu}\right) \mathbf{f} + \left(\epsilon - \frac{1}{\mu}\right) \mathbf{f}^* , \quad 2\mathbf{h}^* = \left(\epsilon + \frac{1}{\mu}\right) \mathbf{f}^* + \left(\epsilon - \frac{1}{\mu}\right) \mathbf{f} . \quad (36)$$

This is a complex form of the constitutive relations (34). It should be noted that eqs. (35) can be resolved under  $\mathbf{f}$ ,  $\mathbf{f}^*$  as well:

$$2\mathbf{f} = \left(\frac{1}{\epsilon} + \mu\right) \mathbf{h} + \left(\frac{1}{\epsilon} - \mu\right) \mathbf{h}^* , \quad 2\mathbf{f}^* = \left(\frac{1}{\epsilon} + \mu\right) \mathbf{h}^* + \left(\frac{1}{\epsilon} - \mu\right) \mathbf{h} ; \quad (37)$$



these are the same constitutive equations (36) in other form. Now let us take into account the Lorentz transformations:

$$\mathbf{f}' = O \mathbf{f} , \quad \mathbf{f}'^* = O^* \mathbf{f}^* , \quad \mathbf{h}' = O \mathbf{h} , \quad \mathbf{h}'^* = O^* \mathbf{h}^* ;$$

then eqs. (35) will become

$$\begin{aligned} \frac{O^{-1} \mathbf{h}' + (O^{-1})^* \mathbf{h}'^*}{2} &= \epsilon \frac{O^{-1} \mathbf{f}' + (O^{-1})^* \mathbf{f}'^*}{2} , \\ \frac{O^{-1} \mathbf{h}' - (O^{-1})^* \mathbf{h}'^*}{2} &= \frac{1}{\mu} \frac{O^{-1} \mathbf{f}' - (O^{-1})^* \mathbf{f}'^*}{2} . \end{aligned}$$

Multiplying both equation by  $O$  and summing (or subtracting) the results we get

$$\begin{aligned} 2\mathbf{h}' &= \left(\epsilon + \frac{1}{\mu}\right) \mathbf{f}' + \left(\epsilon - \frac{1}{\mu}\right) O(O^{-1})^* \mathbf{f}'^* , \\ 2\mathbf{h}'^* &= \left(\epsilon + \frac{1}{\mu}\right) \mathbf{f}'^* + \left(\epsilon - \frac{1}{\mu}\right) O^* O^{-1} \mathbf{f}' . \end{aligned} \quad (38)$$

Analogously, starting from (37) we can produce

$$\begin{aligned} 2\mathbf{f}' &= \left(\frac{1}{\epsilon} + \mu\right) \mathbf{h}' + \left(\frac{1}{\epsilon} - \mu\right) O(O^{-1})^* \mathbf{h}'^* , \\ 2\mathbf{f}'^* &= \left(\frac{1}{\epsilon} + \mu\right) \mathbf{h}'^* + \left(\frac{1}{\epsilon} - \mu\right) O^* O^{-1} \mathbf{h}' . \end{aligned} \quad (39)$$

Equations (38)-(39) represent the constitutive relations after changing the reference frame. In this point one should distinguish between two cases: Euclidean rotation and Lorentzian boosts. Indeed, for any Euclidean rotations

$$O^* = O , \quad \implies \quad O(O^{-1})^* = I , \quad O^* O^{-1} = I ;$$

and therefore eqs. (38)-(39) take the form of (36)-(37); in other words, at Euclidean rotations the constitutive relations do not change their form. However, for any pseudo-Euclidean rotations (Lorentzian boosts)

$$O^* = O^{-1} , \quad \implies \quad O(O^{-1})^* = 0^2 , \quad O^* O^{-1} = O^{*2} ;$$

and eqs. (38)-(39) look

$$\begin{aligned} 2\mathbf{h}' &= \left(\epsilon + \frac{1}{\mu}\right) \mathbf{f}' + \left(\epsilon - \frac{1}{\mu}\right) O^2 \mathbf{f}'^* , \\ 2\mathbf{h}'^* &= \left(\epsilon + \frac{1}{\mu}\right) \mathbf{f}'^* + \left(\epsilon - \frac{1}{\mu}\right) O^2 \mathbf{f}' ; \end{aligned} \quad (40)$$

$$\begin{aligned} 2\mathbf{f}' &= \left(\frac{1}{\epsilon} + \mu\right) \mathbf{h}' + \left(\frac{1}{\epsilon} - \mu\right) O^{*2} \mathbf{h}'^* , \\ 2\mathbf{f}'^* &= \left(\frac{1}{\epsilon} + \mu\right) \mathbf{h}'^* + \left(\frac{1}{\epsilon} - \mu\right) O^{*2} \mathbf{h}' . \end{aligned} \quad (41)$$

In complex 3-vector form these relations seem to be shorter than in real 3-vector form:

$$\begin{aligned}
2\mathbf{D}' &= \epsilon_0 \epsilon \left[ \left( I + \frac{OO + O^*O^*}{2} \right) \mathbf{E}' + \frac{OO - O^*O^*}{2i} c\mathbf{B}' \right] + \\
&\quad + \frac{\epsilon_0}{\mu} \left[ \left( I - \frac{OO + O^*O^*}{2} \right) \mathbf{E}' - \frac{OO - O^*O^*}{2i} c\mathbf{B}' \right], \\
2\mathbf{H}'/c &= \epsilon_0 \epsilon \left[ \left( I - \frac{OO + O^*O^*}{2} \right) c\mathbf{B}' + \frac{OO - O^*O^*}{2i} \mathbf{E}' \right] + \\
&\quad + \frac{\epsilon_0}{\mu} \left[ \left( I + \frac{OO + O^*O^*}{2} \right) c\mathbf{B}' - \frac{OO - O^*O^*}{2i} \mathbf{E}' \right].
\end{aligned} \tag{42}$$

They can be written differently

$$\begin{aligned}
\mathbf{D}' &= \frac{\epsilon_0}{2} \left\{ \left[ \left( \epsilon + \frac{1}{\mu} \right) + \left( \epsilon - \frac{1}{\mu} \right) \text{Re } O^2 \right] \mathbf{E}' + \left( \epsilon - \frac{1}{\mu} \right) \text{Im } O^2 c\mathbf{B}' \right\}, \\
\frac{\mathbf{H}'}{c} &= \frac{\epsilon_0}{2} \left\{ \left[ \left( \epsilon + \frac{1}{\mu} \right) - \left( \epsilon - \frac{1}{\mu} \right) \text{Re } O^2 \right] c\mathbf{B}' + \left( \epsilon - \frac{1}{\mu} \right) \text{Im } O^2 \mathbf{E}' \right\}.
\end{aligned} \tag{43}$$

The matrix  $O^2$  can be presented differently with the help of double angle variable:

$$O^2 = \begin{vmatrix} ch\ 2b + (1 - ch\ 2b) n_1^2 & (1 - ch\ 2b) n_1 n_2 - i sh\ 2b\ n_3 & (1 - ch\ 2b) n_3 n_1 + i sh\ 2b\ n_2 \\ (1 - ch\ 2b) n_1 n_2 + i sh\ 2b\ n_3 & ch\ 2b + (1 - ch\ 2b) n_2^2 & (1 - ch\ 2b) n_2 n_3 - i sh\ 2b\ n_1 \\ (1 - ch\ 2b) n_3 n_1 - i sh\ 2b\ n_2 & (1 - ch\ 2b) n_2 n_3 + i sh\ 2b\ n_1 & ch\ 2b + (1 - ch\ 2b) n_3^2 \end{vmatrix}. \tag{44}$$

The previous result can be easily extended to more generale medias, let us restrict ourselves to linear medias. Indeed, arbitrary linear media is characterized by the following constitutive equations:

$$\mathbf{D} = \epsilon_0 \epsilon(x) \mathbf{E} + \epsilon_0 c \alpha(x) \mathbf{B}, \quad \mathbf{H} = \epsilon_0 c \beta(x) \mathbf{E} + \frac{1}{\mu_0} \mu(x) \mathbf{B}, \tag{45}$$

where  $\epsilon(x), \mu(x), \alpha(x), \beta(x)$  are  $3 \times 3$  dimensionless matrices. Eqs. (45) should be rewritten in terms of complex vectors  $\mathbf{f}, \mathbf{h}$ :

$$\begin{aligned}
\frac{\mathbf{h} + \mathbf{h}^*}{2} &= \epsilon(x) \frac{\mathbf{f} + \mathbf{f}^*}{2} + \alpha(x) \frac{\mathbf{f} - \mathbf{f}^*}{2i}, \\
\frac{\mathbf{h} - \mathbf{h}^*}{2i} &= \beta(x) \frac{\mathbf{f} + \mathbf{f}^*}{2} + \mu(x) \frac{\mathbf{f} - \mathbf{f}^*}{2i}.
\end{aligned} \tag{46}$$

From (45) it follows

$$\begin{aligned}
\mathbf{h} &= [(\epsilon(x) + \mu(x)) + i(\beta(x) - \alpha(x))] \mathbf{f} + [(\epsilon(x) - \mu(x)) + i(\beta(x) + \alpha(x))] \mathbf{f}^*, \\
\mathbf{h}^* &= [(\epsilon(x) + \mu(x)) - i(\beta(x) - \alpha(x))] \mathbf{f}^* + [(\epsilon(x) - \mu(x)) - i(\beta(x) + \alpha(x))] \mathbf{f}.
\end{aligned} \tag{47}$$

Under Lorentz transformations, relations (47) will take the form

$$\begin{aligned}
O^{-1} \mathbf{h}' &= [(\epsilon(x) + \mu(x)) + i(\beta(x) - \alpha(x))] O^{-1} \mathbf{f}' + \\
&\quad [(\epsilon(x) - \mu(x)) + i(\beta(x) + \alpha(x))] (O^{-1})^* \mathbf{f}'^*, \\
(O^{-1})^* \mathbf{h}'^* &= [(\epsilon(x) + \mu(x)) - i(\beta(x) - \alpha(x))] (O^{-1})^* \mathbf{f}'^* + \\
&\quad [(\epsilon(x) - \mu(x)) - i(\beta(x) + \alpha(x))] (O^{-1}) \mathbf{f}',
\end{aligned}$$

or

$$\begin{aligned}\mathbf{h}' &= \epsilon_0 [(\epsilon(x) + \mu(x)) + i(\beta(x) - \alpha(x))] \mathbf{f}' + [(\epsilon(x) - \mu(x)) + i(\beta(x) + \alpha(x))] [O(O^{-1})^*] \mathbf{f}'^*, \\ \mathbf{h}'^* &= \epsilon_0 [(\epsilon(x) + \mu(x)) - i(\beta(x) - \alpha(x))] \mathbf{f}'^* + [(\epsilon(x) - \mu(x)) - i(\beta(x) + \alpha(x))] [O^*(O^{-1})] \mathbf{f}'.\end{aligned}$$

For Euclidean rotation, the constitutive relations preserve their form. For Lorentz boosts we have

$$\begin{aligned}\mathbf{h}' &= [(\epsilon(x) + \mu(x)) + i(\beta(x) - \alpha(x))] \mathbf{f}' + [(\epsilon(x) - \mu(x)) + i(\beta(x) + \alpha(x))] O^2 \mathbf{f}'^*, \\ \mathbf{h}'^* &= [(\epsilon(x) + \mu(x)) - i(\beta(x) - \alpha(x))] \mathbf{f}'^* + [(\epsilon(x) - \mu(x)) - i(\beta(x) + \alpha(x))] O^{*2} \mathbf{f}'.\end{aligned}\tag{48}$$

They are the constitutive equations for arbitrary linear medias in a moving reference frame (similar formulas were produced in quaternion formalism in [95]-[100]).

## 6 Symmetry properties of the matrix Maxwell equation in a uniform media

As noted, Maxwell equations in any media can be presented in the matrix form as follows:

$$(-i\partial_0 + \alpha^i \partial_i) M + (-i\partial_0 + \beta^i \partial_i) N = J. \tag{49}$$

We are to study symmetry properties of this equation under complex rotation group  $SO(3, C)$ . The terms with  $\alpha^j$  matrices were examined in Section 1), the terms with  $\beta^j$  matrix is new. We restrict ourselves to demonstrating the Lorentz symmetry of eq. (49) under two simplest transformations.

First, let us consider the Euclidean rotation in the plane (1 – 2); examine additionally only the term with  $\beta$ -matrices:

$$\begin{aligned}S\beta^1 S^{-1} &= \cos a \beta^1 - \sin a \beta^2 = \beta^j O_{j1}, \\ \beta^2 S^{-1} &= \sin a \beta^1 + \cos a \beta^2 = \beta^j O_{j2}, \\ S\beta^3 S^{-1} &= \beta^3 = \beta^j O_{j3}.\end{aligned}\tag{50}$$

Therefore, we conclude that eq. (49) is symmetrical under Euclidean rotations in accordance with the relations

$$\begin{aligned}(-i\partial_0 + S\alpha^i S^{-1} \partial_i) M' + (-i\partial_0 + S\beta^i S^{-1} \partial_i) N' &= +SJ, \implies \\ (-i\partial_0 + \alpha^i \partial_i) M' + (-i\partial_0 + \beta^i \partial_i) N' &= +J'.\end{aligned}\tag{51}$$

For the Lorentz boost in the plane (0 – 3) we have

$$M' = SM, \quad N' = S^* N = S^{-1} N, \quad S^* = S^{-1};$$

and eq. (49) takes the form (note that the additional transformation  $\Delta = \Delta_{(\alpha)}$  is combined in terms of  $\alpha^j$  (see Sec. 2))

$$\Delta_{(\alpha)} S [(-i\partial_0 + \alpha^i \partial_i) S^{-1} M' + (-i\partial_0 + \beta^i \partial_i) S N'] = \Delta S J,$$

or

$$\Delta_{(\alpha)} [(-i\partial_0 + S\alpha^i S^{-1} \partial_i) M' + S^2(-i\partial_0 + S^{-1} \beta^i S \partial_i) N'] = J',$$

and further

$$(-i\partial'_0 + \alpha^i \partial'_i) M' + \Delta_{(\alpha)} S^2 (-i\partial_0 + S^{-1} \beta^i S \partial_i) N' = J' . \quad (52)$$

It remains to prove the relationship

$$\Delta_{(\alpha)} S^2 (-i\partial_0 + S^{-1} \beta^i S \partial_i) N' = (-i\partial'_0 + \beta^i \partial'_i) N' . \quad (53)$$

By simplicity reason one may expect two identities:

$$\Delta_{(\alpha)} S^2 = \Delta_{(\beta)} \quad \Longleftrightarrow \quad \Delta_{(\alpha)} S = \Delta_{(\beta)} S^{-1} , \quad (54)$$

and

$$\Delta_{(\beta)} (-i\partial_0 + S^{-1} \beta^i S \partial_i) N' = (-i\partial'_0 + \beta^i \partial'_i) N' . \quad (55)$$

Let us prove them for a Lorentzian boost in the plane 0 – 3:

$$S = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & ch\ b & -i\ sh\ b & 0 \\ 0 & i\ sh\ b & ch\ b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} , \quad S^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & ch\ b & -i\ sh\ b & 0 \\ 0 & i\ sh\ b & ch\ b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} ;$$

we readily get

$$\begin{aligned} S^{-1} \beta^1 S &= ch\ b\ \beta^1 - i\ sh\ b\ \beta^2 = \beta^j O_{j1}^{-1} , \\ S^{-1} \beta^2 S &= i\ sh\ b\ \beta^1 + ch\ b\ \beta^2 = \beta^j O_{j2}^{-1} , \quad S^{-1} \beta^3 S = \beta^3 = \beta^j O_{j3}^{-1} . \end{aligned} \quad (56)$$

To verify identity  $\Delta_{(\alpha)} S = \Delta_{(\beta)} S^{-1}$  , or

$$(ch\ b - ish\ b\ \alpha^3) S = (ch\ b - ish\ b\ \beta^3) S^{-1} ,$$

let us calculate separately the left and right parts:

$$(ch\ b - ish\ b\ \alpha^3) S = (ch\ b - ish\ b\ \beta^3) S^{-1} = \begin{vmatrix} ch\ b & 0 & 0 & -i\ sh\ b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\ sh\ b & 0 & 0 & ch\ b \end{vmatrix} .$$

they coincide with each other, so eq. (54) holds. It remains to prove relation (55). Allowing for the properties of  $\beta$ -matrices

$$(\beta^0)^2 = -I, \quad (\beta^1)^2 = -I, \dots \quad \beta^1 \beta^2 = -\beta^3, \quad \beta^2 \beta^1 = +\beta^3 \dots$$

we readily find

$$\begin{aligned} \Delta_{(\beta)} (-i\partial_0 + S^{-1} \beta^i S \partial_i) N' &= (ch\ b - ish\ b\ \beta^3) [ -i\partial_0 + \beta^3 \partial_3 + \\ &\quad + (ch\ b\ \beta^1 - i\ sh\ b\ \beta^2) \partial_1 + (i\ sh\ b\ \beta^1 + ch\ b\ \beta^2) \partial_2 ] N' = \\ &= [ -i(ch\ b\ \partial_0 - sh\ b\ \partial_3) + \beta^3 (-sh\ b\ \partial_0 + ch\ b\ \partial_3) + \beta^1 \partial_1 + \beta^2 \partial_2 ] N' , \end{aligned}$$

that is

$$\Delta_{(\beta)} (-i\partial_0 + S^{-1} \beta^i S \partial_i) N' = (-i\partial'_0 + \beta^1 \partial_1 + \beta^2 \partial_2 + \beta^3 \partial'_3) N' ; \quad (57)$$

the relation (55) holds. Thus, the symmetry of the matrix Maxwell equation in media under the Lorentz group is proved.

## 7 Maxwell theory, Dirac matrices and electromagnetic 4-vectors

Let us shortly discuss two points relevant to the above matrix formulation of the Maxwell theory.

First, let write down explicit form for Dirac matrices in spinor basis:

$$\gamma^0 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix},$$

$$\gamma^1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \gamma^2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \gamma^3 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}.$$

Taking in mind expressions for  $\alpha^i, \beta^i$ , we immediately see the identities

$$\begin{aligned} \alpha^1 &= i\gamma^0\gamma^2, & \alpha^2 &= \gamma^0\gamma^5, & \alpha^3 &= i\gamma^5\gamma^2, \\ \beta^1 &= -\gamma^3\gamma^1, & \beta^2 &= -\gamma^3, & \beta^3 &= -\gamma^1 \end{aligned} \quad (58)$$

so the Maxwell matrix equation in media takes the form

$$\begin{aligned} &(-i\partial_0 + i\gamma^0\gamma^2\partial_1 + \gamma^0\gamma^5\partial_2 + i\gamma^5\gamma^2\partial_3) M + \\ &+ (-i\partial_0 - \gamma^3\gamma^1\partial_1 - \gamma^3\partial_2 - \gamma^1\partial_3) N = J \end{aligned} \quad (59)$$

This Dirac matrix-based form does not seem to be very useful to apply in the Maxwell theory, it does not prove much similarity with ordinary Dirac equation (though that analogy was often discussed in the literature).

Now starting from electromagnetic 2-tensor and dual to it:

$$\tilde{F}_{\rho\sigma} = \frac{1}{2} \epsilon_{\rho\sigma\alpha\beta} F^{\alpha\beta}, \quad F_{\alpha\beta} = -\frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} \tilde{F}^{\rho\sigma}$$

let us introduce two electromagnetic 4-vectors (below  $u^\alpha$  is any 4-vector that in general must not coincide with 4-velocity)

$$e^\alpha = u_\beta F^{\alpha\beta}, \quad b^\alpha = u_\beta \tilde{F}^{\alpha\beta}, \quad u^\alpha u_\alpha = 1; \quad (60)$$

inverse formulas are

$$\begin{aligned} F^{\alpha\beta} &= (e^\alpha u^\beta - e^\beta u^\alpha) - \epsilon^{\alpha\beta\rho\sigma} b_\rho u_\sigma, \\ \tilde{F}^{\alpha\beta} &= (b^\alpha u^\beta - b^\beta u^\alpha) + \epsilon^{\alpha\beta\rho\sigma} e_\rho u_\sigma. \end{aligned} \quad (61)$$

Such electromagnetic 4-vector are presented always in the literature on the electrodynamics of moving bodies, from the very beginning of relativistic tensor form of electrodynamics – see Minkowski [5], Gordon [11], Mandel'stam – Tamm [12]-[13]-[14]; for instance see Yépez – Brito – Vargas [97]. The interest to these field variables gets renewed after Esposito paper [108] in 1998.

In 3-dimensional notation

$$E^1 = -E_1 = F^{10}, \quad cB^1 = cB_1 = \tilde{F}^{10} = -F_{23}, \quad \text{and so on}$$

the formulas (60) take the form

$$\begin{aligned} e^0 &= \mathbf{u} \cdot \mathbf{E} , & \mathbf{e} &= u^0 \mathbf{E} + c \mathbf{u} \times \mathbf{B} , \\ b^0 &= c \mathbf{u} \cdot \mathbf{B} , & \mathbf{b} &= c u^0 \mathbf{B} - \mathbf{u} \times \mathbf{E} , \end{aligned} \quad (62)$$

or in symbolical form

$$(e, b) = U(u) (\mathbf{E}, \mathbf{B}) ;$$

and inverse the formulas (61) look

$$\begin{aligned} \mathbf{E} &= \mathbf{e} u^0 - e^0 \mathbf{u} + \mathbf{b} \times \mathbf{u} , \\ c \mathbf{B} &= \mathbf{b} u^0 - b^0 \mathbf{u} - \mathbf{e} \times \mathbf{u} . \end{aligned} \quad (63)$$

or in symbolical form

$$(\mathbf{E}, \mathbf{B}) = U^{-1}(u) (e, b) ;$$

The relationships can be checked by direct calculation:

$$\begin{aligned} \mathbf{E} &= (u^0 \mathbf{E} + c \mathbf{u} \times \mathbf{B}) u^0 - (\mathbf{u} \cdot \mathbf{E}) \mathbf{u} + (c u^0 \mathbf{B} - \mathbf{u} \times \mathbf{E}) \times \mathbf{u} = \\ &= u^0 u^0 \mathbf{E} + c u^0 (\mathbf{u} \times \mathbf{B}) - (\mathbf{u} \cdot \mathbf{E}) \mathbf{u} + c u^0 (\mathbf{B} \times \mathbf{u}) + \mathbf{u} \times (\mathbf{u} \times \mathbf{E}) = (u^0 u^0 - \mathbf{u} \cdot \mathbf{u}) \mathbf{E} = \mathbf{E} , \\ c \mathbf{B} &= (c u^0 \mathbf{B} - \mathbf{u} \times \mathbf{E}) u^0 - (c \mathbf{u} \cdot \mathbf{B}) \mathbf{u} - (u^0 \mathbf{E} + c \mathbf{u} \times \mathbf{B}) \times \mathbf{u} = \\ &= c u^0 u^0 \mathbf{B} - u^0 (\mathbf{u} \times \mathbf{E}) - c (\mathbf{u} \cdot \mathbf{B}) \mathbf{u} - u^0 (\mathbf{E} \times \mathbf{u}) + c \mathbf{u} \times (\mathbf{u} \times \mathbf{B}) = c (u^0 u^0 - \mathbf{u} \cdot \mathbf{u}) \mathbf{B} = c \mathbf{B} . \end{aligned}$$

The above possibility is often used to produce a special form of the Maxwell equations. For simplicity, let us consider the vacuum case:

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 , \quad \partial_\alpha F^{\alpha\beta} = \epsilon_0^{-1} j^\beta$$

or differently with the help of the dual tensor:

$$\partial_\beta \tilde{F}^{\beta\alpha} = 0 , \quad \partial_\alpha F^{\alpha\beta} = \epsilon_0^{-1} j^\beta \quad (64)$$

They can be transformed to variables  $b^\alpha, b^\alpha$ :

$$\begin{aligned} \partial_\alpha (b^\alpha u^\beta - b^\beta u^\alpha + \epsilon^{\alpha\beta\rho\sigma} e_\rho u_\sigma) &= 0 , \\ \partial_\alpha (e^\alpha u^\beta - e^\beta u^\alpha - \epsilon^{\alpha\beta\rho\sigma} b_\rho u_\sigma) &= \epsilon_0^{-1} j^\beta . \end{aligned}$$

They can be combined into equations for complex field function

$$\Phi^\alpha = e^\alpha + i b^\alpha , \quad \partial_\alpha [ \Phi^\alpha u^\beta - \Phi^\beta u^\alpha + i \epsilon^{\alpha\beta\rho\sigma} \Phi_\rho u_\sigma ] = \epsilon_0^{-1} j^\beta$$

or differently

$$\partial_\alpha [ \delta_\gamma^\alpha u^\beta - \delta_\gamma^\beta u^\alpha + i \epsilon^{\alpha\beta\rho\sigma} g_{\rho\gamma} u_\sigma ] \Phi^\gamma = \epsilon_0^{-1} j^\beta \quad (65)$$

This is Esposito's representation [108] of the Maxwell equations. One may introduce four matrices, functions of 4-vector  $u$ :

$$(\Gamma^\alpha)^\beta_\gamma = \delta_\gamma^\alpha u^\beta - \delta_\gamma^\beta u^\alpha + i \epsilon^{\alpha\beta\rho\sigma} g_{\rho\gamma} u_\sigma , \quad (66)$$

then eq. (65) becomes

$$\partial_\alpha (\Gamma^\alpha)^\beta_\gamma \Phi^\gamma = \epsilon_0^{-1} j^\beta, \quad \text{or} \quad \Gamma^\alpha \partial_\alpha \Phi = \epsilon_0^{-1} j. \quad (67)$$

In the 'rest reference frame' when  $u^\alpha = (1, 0, 0, 0)$ , the matrices  $\Gamma^\alpha$  become simpler and  $\Phi = \Psi$ :

$$\Gamma^0 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \Gamma^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}, \Gamma^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix}, \Gamma^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \quad (68)$$

and eq. (67) takes the form

$$\begin{vmatrix} 0 & \partial_1 & \partial_2 & \partial_3 \\ 0 & -\partial_0 & i\partial_3 & -i\partial_2 \\ 0 & -i\partial_3 & -\partial_0 & i\partial_1 \\ 0 & i\partial_2 & -i\partial_1 & -\partial_0 \end{vmatrix} \begin{vmatrix} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + icB^3 \end{vmatrix} = \epsilon_0^{-1} \begin{vmatrix} \rho \\ j^1 \\ j^2 \\ j^3 \end{vmatrix}, \quad (69)$$

or

$$\text{div} (\mathbf{E} + ic\mathbf{B}) = \epsilon_0^{-1} \rho, \quad -\partial_0 (\mathbf{E} + ic\mathbf{B}) - i \text{rot} (\mathbf{E} + ic\mathbf{B}) = \epsilon_0^{-1} \mathbf{j}. \quad (70)$$

From whence we get equations

$$\begin{aligned} \text{div } c\mathbf{B} &= 0, \quad \text{rot } \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct}, \\ \text{div } \mathbf{E} &= \frac{\rho}{\epsilon_0}, \quad \text{rot } c\mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial ct}, \end{aligned}$$

which coincides with eqs. (2). Relation (68) corresponds to a special choice in (5):

$$\begin{aligned} (-i\alpha^0 \partial_0 + \alpha^j \partial_j) \Psi &= \frac{1}{\epsilon_0} \begin{vmatrix} \rho \\ ij^1 \\ ij^2 \\ ij^3 \end{vmatrix}, \quad \Psi = \begin{vmatrix} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + icB^3 \end{vmatrix}, \\ \alpha^0 &= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}; \end{aligned} \quad (71)$$

and relations (68) – (69) are referred to (71) through identities

$$\beta (-i\alpha^0) = \Gamma^0, \quad \beta \alpha^j = \gamma^j, \quad \text{where} \quad \beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{vmatrix}. \quad (72)$$

Esposito's representation of the Maxwell equation at any 4-vector  $u^\alpha$  can be easily related to the matrix equation of Riemann – Silberstein – Majorana – Oppenheimer in the form (71):

$$(-i\alpha^0\partial_0 + \alpha^j\partial_j)\Psi = J, \quad \implies \quad (73)$$

$$\begin{aligned} (-i\alpha^0\partial_0 + \alpha^j\partial_j)U^{-1}(U\Psi) &= J, \\ -i\alpha^0 U^{-1} &= \beta \Gamma^0, \quad \alpha^j U^{-1} = \beta \Gamma^j, \quad U\Psi = \Phi, \\ \beta (\Gamma^0\partial_0 + \Gamma^j\partial_j) \Phi &= J, \quad \beta^{-1}J = \epsilon_0^{-1}(j^a), \\ (\Gamma^0\partial_0 + \Gamma^j\partial_j)\Phi &= \epsilon_0^{-1} j. \end{aligned} \quad (74)$$

Eq. (74) is a matrix representation of the Maxwell equations in Esposito's form

$$\partial_\alpha [\delta_\gamma^\alpha u^\beta - \delta_\gamma^\beta u^\alpha + i\epsilon^{\alpha\beta\rho\sigma} g_{\rho\gamma} u_\sigma] \Phi^\gamma = \epsilon_0^{-1} j^\beta. \quad (75)$$

Evidently, eqs. (73) and (74) are equivalent to each other. There is no ground to consider the form (74) – (75) obtained through the trivial use of identity  $I = U^{-1}(u)U(u)$  as having certain especially profound sense. Our point of view contrasts with the claim by Ivezić [113]-[118]-[114]-[115]-[119]-[124]-[125]-[126]-[127] that eq. (75) has a status of a true Maxwell equation in a moving reference frame (at this  $u^\alpha$  is identified with 4-velocity).

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